# Schwarzschild spacetime thermodynamics

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2023

**Abstract**

Bardeen, Carter, and Hawking presented the four laws of black hole thermodynamics fifty years ago, in 1973. Only 45 years later, in 2018, Wang and Braunstein rigorously demonstrated that surfaces away from horizons are not thermodynamic, unless they are spherically symmetric. This entails that the laws of black hole thermodynamics can only be generalized to surfaces that are concentric with black hole horizons. Assuming the Schwarzschild metric, this generalization is achieved with a twist: By equating two temperature definitions that should simultaneously hold for an equilibrium thermodynamics of spacetime's degrees of freedom, it is immediately found that the latter only applies to surfaces with constant Newtonian gravitational potential. The laws of thermodynamics can correspondingly be rephrased in terms of Schwarzschild spacetime's state variables at these surfaces.

## 1 Introduction

The laws of thermodynamics outline the constraints that systems obey in their statistical behaviour, usually in terms of state variables and for (quasi) equilibrium states. They can be summarized as (0) transitivity of thermal equilibrium; (1) energy conservation; (2) entropy increase with time; and (3) non-zero temperature. A thermodynamics of (curved) spacetime is thus obtained by defining an energy $E$, temperature $T$, and entropy $S$ for spacetime's relevant degrees of freedom (d.o.f.) $N$, that are in agreement with common thermodynamic concepts and compliant with the above laws, under the assumption of (at least local) equilibrium.

In 1973, Bardeen, Carter, and Hawking presented the "four laws of black hole mechanics" [1]: (0) The surface gravity $\kappa$ of a stationary black hole is constant over the event horizon. (1) Any two neighboring stationary axisymmetric solutions containing [...] a central black hole [with mass $M$ and area $A$ of the event horizon] are related by $\delta M = (\kappa/8\pi)\delta A$ [in Planck units, and neglecting changes in angular momentum and electric charge]. (2) The area $A$ of the event horizon of each black hole does not decrease with time, i.e. $\delta A \geq 0$. (3)

It is impossible by any procedure, no matter how idealized, to reduce $\kappa$ to zero by a finite sequence of operations.

Already in the abstract, the authors pointed out the close resemblance of these laws with the four laws of thermodynamics [1]: "Two of the quantities which appear in these expressions, namely the area $A$ of the event horizon and the "surface gravity" $\kappa$ of the black hole, have a close analogy with entropy and temperature respectively. This analogy suggests the formulation of four laws of black hole mechanics which correspond to and in some ways transcend the four laws of thermodynamics."

Ever since, it has become clear that the laws of thermodynamics should not be limited to black hole horizons, but apply more broadly to the dynamics of spacetime as described by general relativity theory. Indications in this direction have come from, amongst others, the Tolman-Ehrenfest effect [2], the Fulling-Davies-Unruh effect [3], Jacobson's equation of state [4], entropic gravity approaches [5, 6], and the plethora of papers that is being published discussing variations on black hole thermodynamics. Additionally, for theories of (quantum) gravity involving a minimal length scale, some thermodynamic description for the microscopic constituents of spacetime simply must exist [7, 8]. Up to now, however, how exactly these findings are related has not been fully understood, and therefore the four laws of Schwarzschild spacetime thermodynamics have not yet been explicited.

In this work, spacetime's state variables and the thermodynamic relations between them are identified first. By assuming thermal equilibrium for those spacetime degrees of freedom that obey the Davies-Unruh temperature, the recent finding by Wang & Braunstein [9] that surfaces away from horizons are not thermodynamic unless they are spherically symmetric, is reproduced for Schwarzschild spacetimes in Section 2. The four laws of spacetime thermodynamics are formulated therefrom in Section 3, as a rather straightforward generalisation of black hole thermodynamics, and followed by the results discussion in Section 4.

# 2 State variables

## 2.1 Three temperature definitions

In order to uniquely determine the thermodynamic state variables $E$, $T$, $S$, and $N$ in a dynamic spacetime context, one would need a system of at least four independent equations relating these quantities to each other as well as to general relativity theory. One of these equations has to be the connection between temperature and entropy as conjugate thermodynamic variables. Or, the conjugate temperature $T_c$ is defined as a system's energy change with entropy:

$$T_c = \frac{\partial E}{\partial S} \tag{1}$$

The explicit connection with spacetime dynamics can be made through the Davies-Unruh temperature, being the effective temperature $T_a$ that is experienced by an uniformly accelerating observer in a quantized vacuum field [3]:

$$T_a = \frac{\hbar a}{2\pi c k_B} \tag{2}$$

with $a$ the acceleration, and where $\hbar$ is the reduced Planck constant, $c$ the speed of light in vacuum, and $k_B$ the Boltzmann constant. If, through the equivalence principle, this acceleration is due to a gravitational field from a distant mass $M$ (i.e., spacetime curvature induced by $M$), then a connection between Eqs. (1) and (2) can be made using Einstein's mass-energy relation:

$$E = Mc^2 = N_m m_1 c^2 \tag{3}$$

where the latter equation additionally introduces the system's energetic degrees of freedom. It expresses that $M$ is characterized by $N_m$ d.o.f., which each represent a unit-mass $m_1$ that is to be determined later on.

This brings us to three equations for four unknowns. One can however proceed by explicitly imposing an *equilibrium* thermodynamics, at the small price of not knowing beforehand to which spacetime degrees of freedom, i.e., which spatiotemporal 'box', this equilibrium applies. For systems in thermal equilibrium, the energy is equally distributed over the d.o.f. $N_e$ that, from Eq. (1), have to be proportional to $S$. Correspondingly, the equipartition temperature $T_e$ provides a measure for the energy per degree of freedom, given by:

$$T_e = \frac{2E}{k_B N_e} \tag{4}$$

By equating the above temperature expressions, $T_c \equiv T_a \equiv T_e$, one can hence identify whether or when (under what conditions, or for which 'box') an equilibrium thermodynamics of spacetime is valid. This validity is expressed in terms of constraints on the d.o.f., in order to answer our key question: Which spacetime degrees of freedom, if any, would obey Eqs. (1) to (4) simultaneously?

## 2.2 The black hole solution

The answer to the above question is surprisingly straightforward. Connecting Eqs. (2) to (4) for a gravitational pull $g$ from $M = E/c^2$ in presumed thermodynamic equilibrium yields:

$$g = \frac{4\pi GM}{N_e l_P^2} \tag{5}$$

given that $l_P^2 = \hbar G/c^3$ defines the Planck length $l_P$. This expression exactly corresponds to the Newtonian gravitational pull $g_N = 4\pi GM/A$ at a spherical surface $A$ that is centered at the mass centre of $M$, if $A$ represents $N_e$ d.o.f. of two-dimensional size $l_P^2$.

It is clear, however, that Eq. (5) only holds for a specific spherical surface, where the number of d.o.f. is maximized if $l_P$ is their minimum length scale. Taking this surface to be the reference surface of the Schwarzschild metric [10] (also see next subsection), one can identify Eq. (5) with the black hole surface gravity $\kappa = c^4/4GM$ (or identify the radius of $A$ with the Schwarzschild radius $R_S = 2GM/c^2$), resulting in the following:

$$N_e = 4\pi N_m^2 \tag{6}$$

if the unit mass equals half the Planck mass: $m_1 = m_P/2$, with $m_P = (c^2/G)l_P$, which has been suggested before from dimensional arguments [11] and is additionally motivated in the Discussion section. The energetic d.o.f. $N_m$ are thus proportional to the (Schwarzschild) radius, as is known for the Schwarzschild solution to Einstein's field equations. By insertion of the equilibrium surface d.o.f. of Eq. (6) into Eq. (4) (or insertion of $\kappa$ into Eq. (2)), one then obtains:

$$T_{BH} = \frac{\hbar c^3}{8\pi k_B G M} \tag{7}$$

as the surface temperature for the black hole horizon enclosing $M$, also known as the Hawking temperature [12], and confirming our choice for the reference surface. Finally, equating Eq. (1) with Eq. (7) yields:

$$\partial S = 2\pi k_B N_m \partial N_m \tag{8}$$

or by integration on the surface $A$, up to a constant that is set to zero:

$$S_{BH} = \pi k_B N_m^2 = \frac{k_B N_e}{4} \tag{9}$$

as an expression for the equilibrium spacetime entropy of the Schwarzschild horizon, which is proportional to $N_e$ as anticipated. As the spacetime d.o.f. within this surface do not contribute to the black hole thermodynamics, this expression moreover "holographically" holds for the black hole as a whole [13, 14, 15].

### 2.3 Generalization to concentric surfaces

Especially Eq. (7), containing only physical constants except for the mass $M$, suggests that a generalization of black hole thermodynamics to the entire Schwarzschild spacetime must come with a generalization of the mass concept. To that end, one can proceed as in the previous subsections, while thinking of the acceleration in Eq. (2) as the opposite of the acceleration that is required to hold a test mass stationary at constant distance $R > R_S$ from $M$ in the Schwarzschild metric:

$$g = \frac{4\pi G M}{N_e l_P^2 \sqrt{g_{tt}}} \tag{10}$$

with the square root of $g_{tt} = 1 - R_S/R$ being the redshift factor at distance $R$ as the norm of the timelike Killing vector field. This approach is actually identical to adopting the Komar mass definition $M_K = M/\sqrt{g_{tt}}$ [16] in Eq. (5), as if providing a correction factor to the energetic d.o.f. $N_m$ for surfaces away from black hole horizons.

By insertion of $M_K$ into Eq. (4), one obtains:

$$T_A = \frac{\hbar c^3}{8\pi k_B G M \sqrt{g_{tt}}} = \frac{T_{BH}}{\sqrt{g_{tt}}} \tag{11}$$

as the spacetime equilibrium temperature for each concentric spherical surface around $M$ (with $R > R_S$). The temperature of curved spacetime as such represents the black hole surface temperature as experienced by an observer at finite distance $R$ instead of infinity: Eq. (11) equals a distance-dependent mass-reciprocal, while $T_A\sqrt{g_{tt}} = T_A(R = \infty) = T_{BH}$ is constant for all spherical surfaces enclosing $M$, in agreement with the Tolman-Ehrenfest effect [2, 17]. Correspondingly, $T_A$ also expresses the rate of the local, gravitationally-delayed time with respect to proper time, and thus allows introducing the concept of thermal time [18].

Remarkably, equating Eqs. (1) and (4) for the Komar mass definition brings the redshift factor $\sqrt{g_{tt}}$ both in the nominator and denominator of the differential entropy $\partial S$, leaving Eqs. (8) and (9) eventually unchanged for a fixed radial distance:

$$S_A = S_{BH} = \frac{k_B N_A}{4} \tag{12}$$

with the surface equilibrium d.o.f. $N_A = 4\pi R_S^2/l_P^2$ being identical for all spherical surfaces concentric with the Schwarzschild horizon, here explicitly without their interiors. As hence the d.o.f. density decreases quadratically with radial distance, this result may have been expected for gravity in three-dimensional space, although it also means that the holographic principle for the black hole entropy does not generalize to surfaces with $R > R_S$.

# 3 Thermodynamics

Combining the above results, the laws of equilibrium thermodynamics can be given a proper interpretation for the Planck-scale degrees of freedom of Schwarzschild spacetime, each representing a mass $m_P/2$:

(0) The temperature of spacetime, as a local gravity measure, is constant only on mass-centered spherical surfaces. Stated differently, non-spherically-symmetric mass distributions are not in thermal equilibrium (yet).

(1) Spacetime d.o.f. are conserved. Energy changes result from changes in surface d.o.f. (not taking into account changes in angular momentum and electric charge): $\partial E = (k_B T_A/4)\partial N_A$ from Eqs. (1) and (12).

(2) The surface entropy or number of surface d.o.f. cannot decrease: $\partial N_A/\partial t \geq 0$. This inequality expresses that gravity is an aggregating force to spacetime's d.o.f. Stated differently, spacetime cannot induce anti-gravitational effects (in contrast with other forces, cf. Hawking radiation). A different interpretation is that the passage of time relates with the accretion of spacetime d.o.f. [19].

(3) The surface temperature $T_A$ cannot vanish: $T_A > 0$. Given Eq. (11), this statement covers several considerations. First, one cannot have spacetime without gravity, or, spacetime contains energy. Second, the d.o.f. $N_m$ must be finite within any finite radius, as they have been assumed to have a finite spatial extent $l_P$. Finally, any measured (Komar or other) mass must be finite too, also reflecting that masses cannot be accelerated up to or beyond the speed of light.

# 4 Discussion

The above laws obviously reduce to black hole thermodynamics for $N_m = R/l_P$ or, equivalently, $\sqrt{g_{tt}}$ equal to unity. In their more general form, the laws of Schwarzschild spacetime thermodynamics elaborate on Jacobson's expression of the Einstein field equations as an equation of state [4]. One has to conclude that, through these field equations, the dynamics of spacetime is the only one capable of maintaining temperature gradients in thermal equilibrium states without violating the laws of thermodynamics [20], but it can only do so along the radius of spherically-symmetric scenarios. The proportionality between the energetic d.o.f. $N_m$ and the radius $R$ of $A$ moreover indicates that the thermodynamics of the surface rather behaves like that of a one-dimensional quantum system, as has been argued before [21].

Given the laws of black hole thermodynamics, their generalization in the previous section actually should not surprise. If a black hole surface is perceived to have a finite temperature $T_{BH}$ by an observer at infinity, then every intermediate observer at a finite distance $R$ from the Schwarzschild horizon ($R_S < R < \infty$) must perceive a temperature $T_{BH}/\sqrt{g_{tt}}$ due to gravitational redshift. From spherical symmetry, this then has to be valid for the entire surface $4\pi R^2$. Alternatively, the observer at infinity can consider each surface concentric with the Schwarzschild horizon to be a source of thermal radiation with the same temperature $T_{BH}$ (essentially the Tolman-Ehrenfest effect). The radial temperature gradient (as described in the previous paragraph) is then merely an effect to local observers due to gravitational redshift.

Note that one could in principle consider other mass definitions as well for the generalization in Section 2.3, but the result will always be limited to spacetime volumes with constant Newtonian gravitational potential, corresponding to spherical surfaces $A$ centered at the mass centre of $M$, where the gravitational acceleration and the surface normal are parallel. This result was only rigorously proven by Wang & Braunstein in 2018 [9], although it had been suggested before [22, 23]. As noted in these references, this has its consequences for entropic views on gravity. These assume the energetic equipartition relation in Eq. (4) for their d.o.f., and therefore should only apply to spherically-symmetric scenarios according to the above. This result complies with some of the concerns that were raised on entropic gravity theory immediately after Verlinde's publication [6], most convincingly by Visser [22] and Gao [24]. The work presented here in fact supports the latter's reverse conclusion that "the entropy increase [on the surface] is not caused by its statistical tendency to increase entropy as required by the existence of an entropic force, but in fact caused by gravity."

One major question remains for the laws of Schwarzschild spacetime thermodynamics (although not essential to their formulation): What exactly are the degrees of freedom that constitute spacetime? In the very recent literature on emergent spacetime and gravity, there is a tendency to identify them with Standard Model gravitons [25, 26, 27]. For massless gravitons, the speed of gravity indeed matches the speed of light. The attribution of a unit mass to spacetime's d.o.f., as in Eq. (3), then seems contradictory, but is not, as also pointed out in the last references: The numerous gravitons merely represent the central mass $M$ at a distance $R$ in their instantaneous local distribution, which, for static metrics like the Schwarzschild solution, is statistically constant. The factor two difference between the thermodynamic (d.o.f.) mass and the general-relativistic mass ($m_1 = m_P/2$ here but also reported earlier, see [28, 29] for example) can hence be explained from the massless spin-two graviton having two degrees of freedom.

# Acknowledgments

This work elaborates on an essay written for the Gravity Research Foundation 2023 Awards for Essays on Gravitation, which was awarded an Honorable Mention. The Gravity Research Foundation president George M. Rideout Jr. and Samuel L. Braunstein are acknowledged for encouraging the essay's publication. Special thanks go to Jessica Santiago and anonymous essay reviewers for providing insightful suggestions for improvements. Additional gratitude goes to the organisers of the Golden Wedding of Black Holes and Thermodynamics: An Online Celebration (`https://bht50.github.io/`), where this work was presented and discussed.

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